

# NO UNWANTED UNIVERSALLY BAIRE MORPHISMS

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ABSTRACT. We show that the usual proof that there are no morphisms, whose constituent maps are Borel, between certain challenge-response relations generalizes to show that there are no morphisms whose constituent maps are universally Baire.

## 1. MORPHISMS

Let  $\kappa$  be a cardinal. Recall that a set  $A \subseteq {}^\omega\omega$  is  $\kappa$ -*universally Baire* (see [2]) iff there exist trees  $T, S \subseteq {}^{<\omega}\omega \times {}^{<\omega}\delta$  for some cardinal  $\delta$  such that  $p[T] = A$  and in every forcing extension of  $V$  by a forcing of size  $\leq \kappa$ ,  $p[T] = {}^\omega\omega - p[S]$ . A set  $A \subseteq {}^\omega\omega$  is *universally Baire* iff it is  $\kappa$ -universally Baire for all  $\kappa$ . We make a similar definition for relations on  ${}^\omega\omega$  to be universally Baire. We say that a function is universally Baire iff its graph is. Given a set  $A$  which is  $\kappa$ -universally Baire, witnessed by  $T$  and  $S$ , and given a forcing of size  $\leq \kappa$ , we say that the set  $p[T]$  (as computed in the extension) is what  $A$  *lifts* to.

Let  $\mathcal{R}_1 := \langle {}^\omega\omega, {}^\omega\omega, R_1 \rangle$  and  $\mathcal{R}_2 := \langle {}^\omega\omega, {}^\omega\omega, R_2 \rangle$  be challenge-response relations (so  $R_1, R_2 \subseteq {}^\omega\omega \times {}^\omega\omega$ ). It is natural to ask if there is a *morphism* from the first to the second. That is, a pair  $\langle \phi_-, \phi_+ \rangle$  of functions  $\phi_-, \phi_+ : {}^\omega\omega \rightarrow {}^\omega\omega$  such that

$$(\forall x \in {}^\omega\omega)(\forall y \in {}^\omega\omega) \phi_-(x) R_1 y \Rightarrow x R_2 \phi_+(y).$$

In [1] (Theorem 4.15), a situation is given where there can be no such morphism with *either*  $\phi_-$  or  $\phi_+$  Borel. The goal of this document is to show why in the same situation it is impossible for *both*  $\phi_-$  and  $\phi_+$  to be universally Baire. This leaves open the question of whether one of  $\phi_-$  or  $\phi_+$  could be universally Baire.

Consider a challenge-response relation  $\mathcal{R} = \langle {}^\omega\omega, {}^\omega\omega, R \rangle$  which has an interpretation in every forcing extension (this happens when the relation  $R$  is universally Baire for example). Given a forcing  $\mathbb{P}$ , we say that  $\mathbb{P}$  is  $\mathcal{R}$ -adequate if

$$1_{\mathbb{P}} \Vdash (\forall x \in {}^\omega\omega)(\exists y \in {}^\omega\omega \cap \check{V}) x R y.$$

**Lemma 1.1.** *Let  $\kappa$  be an infinite cardinal. Let  $f : {}^\omega\omega \rightarrow {}^\omega\omega$  be a function whose graph is  $\kappa$ -universally Baire. Then in every forcing extension by a poset of size  $\leq \kappa$ ,  $f$  lifts to a function defined on all of  ${}^\omega\omega$ .*

*Proof.* Let  $\kappa$  be a cardinal and let  $T, S$  be trees witnessing that the graph of  $f$  is  $\kappa$ -universally Baire. Let  $\mathbb{P}$  be a poset of size  $\leq \kappa$ . Let  $G$  be  $(V, \mathbb{P})$ -generic. We want to show that  $(p[T])^{V[G]}$  is the graph of a total function in  $V[G]$ . Given any  $x \in ({}^\omega\omega)^{V[G]}$ , we want some  $y \in ({}^\omega\omega)^{V[G]}$  satisfying  $(x, y) \in p[T]$ .

Towards a contradiction, fix an  $x \in ({}^\omega\omega)^{V[G]}$  such that there is no such corresponding  $y$ . From  $T \subseteq {}^{<\omega}\omega \times {}^{<\omega}\omega \times {}^{<\omega}\delta$  we can form  $T_x \subseteq {}^{<\omega}\omega \times {}^{<\omega}\delta$  which is a well-founded tree. Since  $T_x$  is well-founded, it has some rank function  $\sigma : T_x \rightarrow \omega_1$ . Consider the tree  $W$  whose nodes are pairs consisting of an element of  ${}^{<\omega}\omega$  and a partial attempt to build a rank function for the corresponding tree from  $T$ . We have that  $(x, \sigma)$  is a path through  $W$ . Since  $W \in V$  and  $W$  has a path in  $V[G]$ , it has a path in  $V$ . Such a path witnesses that  $f$  is not total in  $V$ , which is a contradiction.

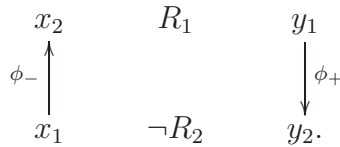
Using similar reasoning, it can be shown that in  $V[G]$  there are not  $x, y_1, y_2$  with  $y_1 \not\leq y_2$  such that  $(x, y_1) \in p[T]$  and  $(x, y_2) \in p[T]$ .  $\square$

We will show the following. The hypothesis arises in practice, for example the proof that there is no Borel morphism from the splitting relation to the domination relation (see [1] Theorem 4.15).

**Proposition 1.2.** *Let  $\mathcal{R}_1 = \langle {}^\omega\omega, {}^\omega\omega, R_1 \rangle$  and  $\mathcal{R}_2 = \langle {}^\omega\omega, {}^\omega\omega, R_2 \rangle$  be challenge-response relations with  $R_1$  and  $R_2$  universally Baire. Suppose there is a forcing  $\mathbb{P}$  which is  $\mathcal{R}_1$ -adequate but not  $\mathcal{R}_2$ -adequate. Then there is no morphism  $\langle \phi_-, \phi_+ \rangle$  from  $\mathcal{R}_1$  to  $\mathcal{R}_2$  such that both the graph of  $\phi_-$  and the graph of  $\phi_+$  are universally Baire.*

*Proof.* Consider a pair of functions  $\langle \phi_-, \phi_+ \rangle$  that are universally Baire. Let  $\Phi$  be the statement that there exist  $x_1, x_2, y_1, y_2 \in {}^\omega\omega$  satisfying the following

- 1)  $\phi_-(x_1) = x_2$ ;
- 2)  $\phi_+(y_1) = y_2$ ;
- 3)  $x_2 R_1 y_1$ ;
- 4)  $\neg x_1 R_2 y_2$ .



We claim that  $\Phi$  is equivalent to a statement which asserts the existence of a path through a tree in the ground model. The conditions 1)-4) are all of this form. For example, if  $R_2 = p[T] = {}^\omega\omega - p[S]$ , then 4) is equivalent to saying that  $(x_1, y_2)$  is a path through  $S$ .

Note that by the definition of a morphism, if  $\Phi$  holds, then  $\langle \phi_-, \phi_+ \rangle$  is not a morphism from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ . Now since  $\Phi$  is equivalent to a statement which asserts the existence of a path through a tree in the ground model, it is absolute between  $V$  and forcing extensions. In particular, if we show that  $\Phi$  holds after forcing with  $\mathbb{P}$ , then we are done.

Force with  $\mathbb{P}$  to get  $V[G]$ . Let  $x_1 \in ({}^\omega\omega)^{V[G]}$  witness that  $\mathbb{P}$  is not  $\mathcal{R}_2$ -adequate. That is, there is no  $y \in {}^\omega\omega \cap V$  satisfying  $x_1 R_2 y$ . By the lemma above, we may speak of  $\phi_-(x_1)$ . Since  $\mathbb{P}$  is  $\mathcal{R}_1$ -adequate, let  $y_1 \in {}^\omega\omega \cap V$  satisfy  $\phi_-(x_1) R_1 y_1$ . By what we said about  $x_1$ , we have  $\neg x_1 R_2 \phi_+(y_1)$ . Hence,  $\Phi$  is satisfied. This completes the proof.  $\square$

Note that this proposition says that  $\phi_-$  and  $\phi_+$  cannot *both* be universally Baire, whereas Theorem 4.15 of [1] says that *neither*  $\phi_-$  nor  $\phi_+$  can be Borel.

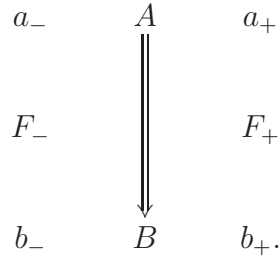
## 2. WEAK MORPHISMS

There is a variant of the notion of morphism which is more general when one does not assume the Axiom of Choice. The idea is to replace functions with multiple valued functions.

**Definition 2.1.** Given challenge-response relations  $\mathcal{A} = \langle {}^\omega\omega, {}^\omega\omega, A \rangle$  and  $\mathcal{B} = \langle {}^\omega\omega, {}^\omega\omega, B \rangle$ , a *weak morphism* from  $\mathcal{A}$  to  $\mathcal{B}$  is a pair  $\langle F_-, F_+ \rangle$  of relations  $F_-, F_+ \subseteq {}^\omega\omega \times {}^\omega\omega$  such that the following are satisfied:

- 1)  $(\forall b_- \in {}^\omega\omega)(\exists a_- \in {}^\omega\omega) b_- F_- a_-$ ;
- 2)  $(\forall a_+ \in {}^\omega\omega)(\exists b_+ \in {}^\omega\omega) a_+ F_+ b_+$ ;
- 3) for all  $a_-, a_+, b_-, b_+ \in {}^\omega\omega$ , if  $b_- F_- a_-$ ,  $a_+ F_+ b_+$ , and  $a_- A a_+$ , then  $b_- B b_+$ .

This definition can be remembered by the following picture.



If there is a morphism from  $\mathcal{A}$  to  $\mathcal{B}$ , then there is a weak morphism from  $\mathcal{A}$  to  $\mathcal{B}$ . Assuming the Axiom of Choice, the other direction holds as well. The proof of Proposition 1.2 can be easily modified to prove the corresponding result for weak morphisms.

#### REFERENCES

- [1] Andreas Blass. Combinatorial Cardinal Characteristics of the Continuum. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory Volume 1*. Springer, New York, NY, 2010.
- [2] Qi Feng, Menachem Magidor, and W. Hugh Woodin. Universally Baire sets of reals. In H. Judah, W. Just, and H. Woodin, editors, *Set Theory of the Continuum*, volume 26 of *Mathematical Sciences Research Institute Publications*, pages 203–242, Heidelberg, 1992. Springer–Verlag.

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